

The prime analog of the Kepler-Bouwkamp constant

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Begin with a circle of radius R_1 and circumscribe it with an equilateral triangle. Circumscribe the triangle with another circle. Let the radius of the second circle be R_2 . Circumscribe the second circle with a square. Continue in this manor, circumscribing with a circle, then a regular pentagon, circle, regular hexagon, *ad infinitum*, each time adding one more side to the regular polygon. The limit of the ratio of the radius of the outer circle to the inner is

$$K = \lim_{n \rightarrow \infty} \frac{R_n}{R_1} = 8.7000366252\dots,$$

which is sometimes referred to as the *polygon circumscribing constant* [1]. Since

$$\frac{R_n}{R_1} = \prod_{k=3}^n \sec\left(\frac{\pi}{k}\right),$$

we have

$$K = \prod_{k=3}^{\infty} \sec\left(\frac{\pi}{k}\right).$$

If, instead of circumscribing, we inscribed at each step we get the *Kepler-Bouwkamp constant* [2], which is

$$\rho = \frac{1}{K} = \prod_{k=3}^{\infty} \cos\left(\frac{\pi}{k}\right) = 0.1149420448\dots$$

What if only regular polygons with prime (greater than 2) number of edges are considered? In other words, what is

$$K_p = \prod_{p \geq 3} \sec\left(\frac{\pi}{p}\right), \quad (1)$$

where p denotes the primes $p = 2, 3, 5, \dots$?

Taking the natural log of both sides of equation (1),

$$\ln(K_p) = \sum_{p \geq 3} \ln\left(\sec\left(\frac{\pi}{p}\right)\right). \quad (2)$$

Expanding the summand,

$$\ln\left(\sec\left(\frac{\pi}{p}\right)\right) = \sum_{k=1}^{\infty} \frac{2^{2k} (2^{2k} - 1) |B_{2k}|}{2k (2k)!} \left(\frac{\pi}{p}\right)^{2k}, \quad (3)$$

where B_n is the n th Bernoulli number, which can be obtained from

$$\frac{x}{\exp(x) - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}.$$

Using the identity

$$B_{2n} = \frac{(-1)^{n-1} 2 (2n)!}{(2\pi)^{2n}} \zeta(2n),$$

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where ζ is the Riemann zeta function, equation (3) becomes

$$\ln \left(\sec \left(\frac{\pi}{p} \right) \right) = \sum_{k=1}^{\infty} \frac{(2^{2k} - 1) \zeta(2k)}{k} \frac{1}{p^{2k}}.$$

Thus, equation (2) can be written as a double sum,

$$\ln(K_p) = \sum_{p \geq 3} \sum_{k=1}^{\infty} \frac{(2^{2k} - 1) \zeta(2k)}{k} \frac{1}{p^{2k}}. \quad (4)$$

Equation (4) is absolutely convergent; the order of summation can be interchanged, *viz.*

$$\ln(K_p) = \sum_{k=1}^{\infty} \frac{(2^{2k} - 1) \zeta(2k)}{k} \sum_{p \geq 3} \frac{1}{p^{2k}}.$$

The summation over the primes can be written in terms of the prime zeta function, which is

$$\mathcal{P}(n) = \sum_p \frac{1}{p^n}.$$

Thus,

$$\ln(K_p) = \sum_{k=1}^{\infty} \frac{(2^{2k} - 1) \zeta(2k)}{k} \left(\mathcal{P}(2k) - \frac{1}{2^{2k}} \right). \quad (5)$$

Perhaps the only unfamiliar term is the prime zeta function, but this can be evaluated in terms of the more familiar Riemann zeta function using the identity [3]

$$\mathcal{P}(n) = \sum_{k=1}^{\infty} \frac{\mu(k) \ln(\zeta(nk))}{k},$$

where μ is the Möbius function.

Equation (5) quickly converges; taking the exponential gives

$$K_p = 3.1965944300 \dots$$

The prime analog of the Kepler-Bouwkamp constant is

$$\rho_p = \frac{1}{K_p} = \prod_{p \geq 3} \cos \left(\frac{\pi}{p} \right) = 0.3128329295 \dots$$

We also have, with no extra work, the following products over the positive nonprimes, $q = 1, 4, 6, \dots$,

$$K_q = \frac{K}{K_p} = \prod_{q \geq 4} \sec \left(\frac{\pi}{q} \right) = 2.7216579443 \dots,$$

and

$$\rho_q = \frac{1}{K_q} = \prod_{q \geq 4} \cos \left(\frac{\pi}{q} \right) = 0.3674231004 \dots$$

[1] N. J. A. Sloane, *On-line encyclopedia of integer sequences*, A051762 and A085365

[2] S. R. Finch, *Mathematical Constants*, Cambridge University Press, 2003

[3] E. C. Titchmarsh and D. R. Heath-Brown, *The Theory of the Riemann Zeta-Function*, 2nd Edition, Oxford University Press, 1986